MA 241 : Ordinary Differential Equations (JAN-APR, 2018)

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Problem set 3

If you notice any mistakes in the problems, kindly bring it to my attention

1. Consider the ODE $\dot{y} = h(t)g(y)$ in the variable separable form and g, h are continuous, $\frac{1}{g}$ also continuous in y. Use change of variable formula to show that
$$\int \frac{1}{g(y)}\, dy = \int h(t)dt.$$

2. Show that, if $\frac{N_t - M_y}{M}$ is a function of y alone, then an IF of the form $\mu = \mu(y)$ can be found for $M + N\dot{y}$.

3. Find the condition under which $M(t,y) + N(t,y)\dot{y} = 0$ has IF μ of the form
   i. μ is a function of a single variable $z = t + y$
   ii. μ is a function of a single variable $z = ty$.
   Write down the IF in each of the cases.

4. Find a solution of $\ddot{y} + 2t\dot{y} + (1 + t^2)y = 0$ of the form $y = uv$.

5. Analyze the boundary value problem $\ddot{y} + ky = 0$ in $[0, 1]$, k is a constant with the boundary condition $y(0) = 0, y(1) = 0$. Using the ODE, show that $\{\sin \pi nt\}_{n\in\mathbb{N}}$ is an orthogonal set in $L^2[0, 1]$. Consider the same ODE with appropriate boundary condition to show the orthogonality of $\{\cos \pi nt\}$, $\{\sin \pi nt, \cos \pi nt\}$.

6. Use the method of variation of parameters to solve i) $\dot{y} = ay + b$ ii) $\dot{y} = \frac{-y + \sin t}{t}$.

7. Show that the equation
$$\frac{y + t\ g(t^2 + y^2)}{t^2 + y^2} + \left(\frac{y\ g(t^2 + y^2) - t}{t^2 + y^2}\right)\dot{y} = 0$$

is exact. In other words $\dfrac{1}{t^2+y^2}$ is an IF for $y + tg(t^2+y^2) + (yg(t^2+y^2) - t)\dot{y} = 0$ for any differentiable function g.

8. Find the general solution of
   i) $t^2\ddot{y} + t\dot{y} - y = 0$
   ii) $t^3\ddot{y} + t^2\dot{y} - ty = 0$
   iii) $\ddot{y} - f(t)\dot{y} + (f(t)-1)y = 0$.

9. Find the general solution of $t^3\ddot{y} + t^2\dot{y} - ty = \dfrac{t}{1+t}$.

10. Recall the Spring mass system equation and solve the problem with external force $F = F_0 \cos \omega t$ for both $\omega \neq \omega_0$ and $\omega = \omega_0$, where $\omega_0 = \sqrt{\dfrac{k}{m}}$ is the natural frequency.

11. Analyze the IVP $\dot{y} = y^2$, $y(t_0) = y_0$ in the domain $R = \{(t, y) : |t - t_0| \leq a, |y - y_0| \leq b\}$ with various a and b. In particular, find the interval of existence given by Peano's existence theorem, maximal interval of existence etc. Why the function y^2 is not Lipschitz in $\mathbb{R}$

12. Prove the following

    (1) The functions $\sin x, \cos x, ax + b$ are all Lipschitz in $\mathbb{R}$.

    (2) The polynomial functions $f_k(x) = x^k$, $k = 1, 2, 3, \cdots$ are all locally Lipschitz in $\mathbb{R}$, but it is not (globally) Lipschitz in $\mathbb{R}$. In fact, it is Lipschitz in any finite interval.

    (3) $f(x) = |x|^{1/2}$ is not locally Lipschitz at 0, that is f not Lipschitz in any interval (a, b) containing the origin. But it is Lipschitz in any interval (finite or infinite) away from the origin. More specifically prove it is Lipschitz in (a, b) if $a > 0$ and it is Lipschitz in (a, b) with $b < 0$.. Is it Lipschitz in $(0, 1)$. Substantiate.

13. Discuss the Lipschitz continuity of the following functions with respect to y.

    (i) $f(t, y) = y^{2/3}$

    (ii) $f(t, y) = \sqrt{|y|}$

    (iii) $f(t, y) = \begin{cases} \dfrac{4t^3 y}{t^4 + y^4} & (t, y) \neq (0, 0) \\ 0 & (t, y) = (0, 0) \end{cases}$

    (iv) $f(t, y) = t|y|$ on $D : |t| \leq a, |y| \leq b$

    (v) $f(t, y) = t \sin y + y \cos t \quad D : |t| \leq a, |y| \leq b$

    (vi) $f(t, y) = y + [t]$ on a bounded domain D in $\mathbb{R}^2$ where $[t]$ denotes the greatest integer less than or equal to t. Note that f is not continuous in t.

14. Study the IVP $\dot{y} = \sin y$, $y(t_0) = y_0$ without solving the problem with different initial time and initial value. Global existence? Sketch various solutions in the same (t, y) plane. Finally, solve the problem explicitly

15. Recall the Population dynamics problem from Problem Set 1. Analyze the problem in the present context (without solving) for existence and uniqueness, global existence etc. for various values of initial time and initial position.